Reasoning and comparing fractions

Potentials of commensuration meaning

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Introduction

Developing a deep understanding of fractions requires a conceptual understanding of different meanings of fractions and relationships between them (Clarke, Roche, & Mitchell, 2008). Sowder (1988) underlined that referring back to models to make comparison decisions can cause additional problems as that students are "model poor". In primary school, fractions are studied as splitting the unit meaning. The French curriculum describes this meaning of fractions as follows: you split a unit by a natural number of equal parts and you take a number of those parts, potentially largest than the numbers of parts included in the unit. Brousseau's work (1987) highlights another original meaning of fraction, named commensuration. However commensuration did not find place in the French curriculum. In this study, we aim at examining the potentials of the commensuration meaning for fraction comparison. In particular, we address the following research questions: how do students use this meaning in comparing fractions? what strategies are students likely to use in such tasks?

Framework & methodology

Commensuration fraction meaning is distant from splitting the unit meaning. In the context of commensuration, $\frac{a}{b}$ is the relation between two physical quantities, such as b times the first quantity gives a times the second one. Classroom experiments related to teaching and learning this meaning of fraction were conducted in primary school, during the eighties (Brousseau, & Brousseau, 1987; Brousseau, 2002; Brousseau, Brousseau, & Warfield, 2014). In particular, Brousseau designed the "thickness of a sheet of paper" situation in which children had to find an optimal measurement strategy in situations of measuring something so thin that previously known techniques of direct measurement could not be applied. The situation involved measuring and comparing the thicknesses of various types of paper. The aim was to characterize the thickness of a sheet first by a pair of whole numbers (for example, 56 sheets – 8 mm) and then by a fraction (for example $\frac{8}{56}$). Thus, the thickness of a sheet is $\frac{8}{56}$ mm means that 56 sheets of paper measure 8 mm. Building on this previous research work, we design and experiment with mathematical tasks based on the motion of robots along a number line intended to promote teaching and learning of the commensuration meaning of fraction with students at grade 6. These tasks contextualize commensuration: some robots make regular bounds on the number line, reaching a natural number a of units in b bounds (reaching also $n \times a$ units in $n \times b$ bounds); $\frac{a}{b}$ is the length of robot's bound. Because of a lack of space, we don't explain similarities and differences with Brousseau's "thickness of a sheet of paper" situation. Three teachers volunteered to participate in our study (each in different schools). A case study approach and qualitative analysis methods were employed: we video recorded all sessions with students and analyzed data (transcriptions related to videos, students' writing works) in order to understand how students used the commensuration meaning of fraction in a set of fraction comparing tasks.

Results

In solving fraction comparison tasks, students build different strategies on the go. They are able to contextualize their strategies and justify their choices by using commensuration meaning of fractions.

Find a common denominator strategy: to compare two fractions $\frac{a}{b}$ and $\frac{c}{d}$ (*i.e.* two lengths of robot's bounds), students looked for a "common number of bounds" which is the common multiple of two numbers corresponding to two given "numbers of bounds" or denominators. Their rationale is as follows: for the same number of bounds, the robot having the smallest bound is the one who covers the smallest distance. In such a reasoning, students need to produce equivalent fractions by reasoning as follows: a robot reaching *a* units in *b* bounds goes *n* times further (and reaches *n* times *a* units) in *n* times more bounds ($n \times b$ bounds). Such a rationale corresponds to the algebraic equivalence of fractions: $\frac{a}{b} = \frac{n \times a}{n \times b}$. We observe that students employ this strategy even for fractions the denominator of which is not a multiple of the other one.

Find a common numerator strategy: in a symmetrical manner, to compare fractions, students use a reasoning based on "a same distance covered by the robots": for the same distance covered by the robots, the robot having the smallest bound is the one who make the most bounds. Both the strategies relying on numerators (equalizing distances covered by robots) and on denominators (equalizing numbers of robots' bounds) have equal chances to appear in classes.

A qualitative strategy: in comparing fractions, another strategy, a priori more unusual, is used by students. It is based on the comparison of numerators on the one hand and on denominators on the other hand: $\frac{a}{b} < \frac{c}{d}$ if $a<c$ and $b>d$. Students use this strategy as follows: this robot makes the smallest bound because with more bounds ($b>d$), it covers a smaller distance ($a<c$).

Conclusion

The experiments conducted in this study underline that students succeed in building and using diverse strategies to compare fractions. Such students' strategies recover rich mathematical arguments by using the context of the situation. This observation illustrates the strong potentials of *commensuration* meaning in order to avoid fractions teaching and learning relying exclusively on formal rules or numerical algorithms. Moreover, reasoning used by students may be extended easily to operations in fraction calculus such as multiplying and dividing fractions by a natural number: the length of a bound which is *n* times smaller / larger of a given bound may be found by thinking that the robot will make *n* times more / less bounds to cover the same distance. New experiments are under way…

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