The (related) unit: a mathematical structure for flexibility and sense

Christine Chambris, *Université de Cergy-Pontoise, LDAR, France*; christine.chambris@u-cergy.fr

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Research questions, objectives and framework

This paper reports a study related to a specific structure of school arithmetic and some of its possible roles in flexibility and sense. The structure is the *related unit* (Chambris, to appear). In previous studies, we demonstrated major changes in France driven by the New Math (1955-1975) – at the level of scholarly knowledge –that impacted school arithmetic (Chambris, 2008, 2017), as well as specific changes regarding place value from 1980s (Chambris, 2008, 2018). The former can be summed up in 'impact on school arithmetic of changes involving magnitudes in academic mathematics: modern knowledge based on sets 'replaced' classical knowledge based on magnitudes', and the latter in 'disappearance of "numeration units" (Chambris, 2008), i.e. tens (T), hundreds (H), etc., in place value school knowledge'. The name stresses that tens, hundreds were *units* in place value school knowledge; it resonates with 'prior to New Math' (PNM) "units of different orders". In the 1980s, the 'modern' theory of place value replaced the 'classical' one. This change implied that numeration units disappeared more or less rapidly from French elementary arithmetic and were replaced by 1, 10, 100, and 1000. The *sense* of the *hundred* as a *unit* might have been lost. Furthermore, PNM and recent Chinese curricula (Ma, 1999) have similarities especially on column calculation (Chambris, 2008), and (Ma, 2013) suggest the *unit* plays a specific role in the Chinese curriculum. What are the units really? What are potentials of units in term of flexibility and sense? Are these potentials available in current French curriculum?

The study is framed by the anthropological theory of didactics (ATD) (Chevallard, 1997). The ATD assumes teaching and learning is an anthropological phenomenon, ruled by institutions. The ATD enlarges the theory is *didactic transposition*: knowledge that is learned in a given institution is framed by knowledge that is taught (by teacher of the institution), which is framed by knowledge to be taught (in written curricula), which is framed by scholarly knowledge (selected by the institution). It introduced the notion of *praxeology*: any human practice can be described in terms of praxis (type of tasks and technique to rule the tasks of a type) and logos (justification –called a technology- and theory that organizes the justifications).

Methodology

Our methodology is to analyze the roles of units in arithmetic praxeologies: first in reference texts for knowledge to be taught in France PNM and in China today (the data are French treatises of 18^{th} and 19^{th} centuries and "Theory of School Arithmetic" (Ma & Kessel, 2018, MK hereafter) for Chinese context), then, a mental calculation teaching episode in a French class.

Results and implications. Relations to the themes chosen.

Both reference texts have a broadly similar structure, introduce number as an accumulation of units, and include concrete and abstract numbers. A *concrete* (resp. *abstract*) *number* is a number whose units are (resp. not) named. Similarly to PNM treatises, place value approach in MK involves ten, hundred, etc. as units. Both definitions of multiplication are based on multiplier and multiplicand ("The *product* of two numbers is a third number which contains as many units as one number taken as many times as the units in the other (…) Multiplicand is the number to be taken. The multiplier is the number that indicates how many times the multiplicand is taken." (MK, p. 452-453)). The role of units is less explicit in PNM texts than in MK. Despite differences, the two texts provide broadly similar technologies for numbers, whole number place value, and the four operations.

The reference texts start differently. The PNM texts start with a definition of magnitudes that introduces a discourse on units: (1) "A unit is an arbitrary value of magnitude which makes comparison possible with other magnitude values of the same kind", whereas MK starts with a definition of units: "A single thing, or one, is called a *unit* or *unit one*. A group of things or a group of units, if considered as a single thing or one, is also called a *unit*, a *unit one* or a *one*". This helps to see that, in the PNM context, drawing on (1): not only 'a centimeter' is a unit but the '3 centimeters' *length* too. We call such type of unit –a unit which is related to another one –a *related unit*. As a 3 cm length is a unit with which 21 cm length measures 7, the number 3 is a unit with which the number 21 measures 7. Similarly, 21 (ones) measure 2 tens and 1 (one).

The mathematics that appears in PNM treatises was elaborated in the 18th century, in the scholarly mathematical paradigm of the time: the 'Idealization of reality paradigm of axiomatization', also 'Euclidean axiomatics'. Mathematics included magnitudes that provided numbers. It was replaced by 'no contradiction paradigm' at the turn of 20th century (Otte, 2007). The former paradigm takes place in the 'theory component' of the PNM praxeologies. We don't know whether it shapes MK.

Rinaldi and Chambris (2019) discussed a classroom episode about 137-50. Two key types of techniques enable to run the task: (1) counting ten by ten decreasingly from 100 (a hard stuff in French), (2) decompose 137, 100 or 107using tens then 13T-5T=8T or 100-20=1H-2T=10T-2T=8T. In terms of technology, (2) requires: 1H = 10T. The most advanced students developed type (2) techniques. In a collective discussion, with limited success, the teacher used (1). Coherently, the lack of numeration units in French context implies 137=13T+7O is taught in a rigid manner: the number of tens (i.e. 13T) is the 'number one reads from the left side of the numeral up to the ten's place' (Chambris, et al., to appear), such technique does not provide flexible decompositions of numbers. Hence, the related units (1H and 1T) (and the relation between both, 1H=10T) appear to be key objects for flexibility in calculation for they provide flexible number decompositions and calculation means, but they are little available in the French context.

To sum up, magnitudes provided a mathematical object –the (related) unit –that disappeared from French reference texts around 1970-1980. Calculation example shows 1) how it potentially provides flexibility and sense and 2) some effects of its current absence in terms of lack of flexibility and sense. Coulange and Train (submitted) explore other facets of related units in fractions sense. Prior to New Math, such units developed in Euclidean paradigm in which numbers emerge from idealization of reality. Fields medalist Thom claimed "Mathematical thought is born of the spirit's need to simulate external reality" (1973, p. 198). Does current state reflect his critic of New Math orientations? Under which conditions could units live and foster sense and flexibility in arithmetic?

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